Real Candidacy Games: A New Model for Strategic Candidacy

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ABSTRACT

We introduce Real Candidacy Games (RCGs)—a novel strategic candidacy model, where candidates have a continuous range of positions that affect their attractiveness for voters. We also allow candidates to have their own non-trivial preferences over the candidate set. We study RCGs with *restricted* and *unrestricted* positioning strategies to establish conditions for Nash Equilibrium (NE) existence. That is, we investigate under what voting rules and tie-breaking schemes, a stable candidate positioning exists. While for several voting rule classes (e.g., Condorcet-Consistent) we obtain positive results, we also show that for some scoring rules there are examples without a NE for an arbitrarily large number of voters.

CCS Concepts

•**Theory of computation** → **Solution concepts in game theory;**
•**Computing methodologies** → *Multi-agent systems;*

Keywords

Candidacy Games, Hotelling-Downs Model, Social Choice

1. INTRODUCTION

Acrimony is often innate to political discourse even within the same region of the political spectrum; recent US presidential primaries have shown this with blunt clarity. In direct political opposition, extreme strategies are not uncommon. A party might sacrifice its own chances, just to prevent a much-disliked opponent from winning. In French regional elections the Socialist Party withdrew from the race, consolidating left-wing voters, and ensuring the success of a centrist candidate against the right-wing National Front.

Strategic behaviour of this type has been captured by Strategic Candidacy Game (SCG)—a formal game theoretic model, where election candidates may have their own preferences over the candidate set, and *abstain* strategically. The basic model was introduced by Dutta et al. [4, 5], with the results later strengthened by Lang et al. [11], and Ehlers and Weymark [6]. SCGs were also extended by additional variations, including iterative voting processes [14], multi-valued election outcomes [8], and probabilistic voting [15].

In this paper, we further SCGs' diversity by allowing a candidate to reposition herself within the political spectrum. This occurs in parliamentary democracies where a smaller party becomes pivotal in forming a government. Such parties may only be interested in benefiting a particular social group. They *lend* their strength in exchange for coalitional guarantees, and align their political agenda accordingly. However, even a well-defined political stance may vary beneficially. For instance, parties at the same end of the political spectrum may make their positions more extreme to distinguish themselves from one another.

To achieve the combined agility of positional fluidity and strategic participation, we merge the Hotelling-Downs model with SCGs. We name the resulting model Real Candidacy Games (RCGs).

The Hotelling-Downs model (HDM) was originally introduced by Hotelling [10] for the problem of ice cream vendors along a stretch of beach, followed by Downs [3], who extended it to ideological positioning in a multipartisan democracy (see also [7]). HDM makes an assumption of a non-atomic voter distribution along the political spectrum. Then, once candidates choose their political stance, voters gravitate towards the closest candidate and vote for her. A natural model for very large plurality elections, HDM has successfully explained several political and economic behaviors—for instance, the median placement policy, which explains why food-vendors clump together and opposing parties agree on laws that benefit neither. Unfortunately, HDM is not guaranteed to produce an equilibrium [13]. However, some refinements produce more favorable conditions for the existence of Nash Equilibria, e.g., the work by Brusco et al. [1] that replaces the Plurality voting rule by Plurality with a run-off. In our RCG model, we build on two specific pieces of research in the Hotelling-Downs literature, namely that of Sengupta and Sengupta [16] and Feldman et al. [9].

The work by Sengupta and Sengupta (abbreviated "S+S") made a step in the direction of SCGs; it includes the possibility of a candidate abstaining. As a result, an equilibrium guarantee is obtained, quite similar to that of lazy-candidate games [12] and lazy-voter games [2]. However, the model of S+S does not encompass the strategic depth of SCGs. In particular, their candidates do not possess a preference order over the candidate set, except for self-support. That is, S+S assume that it is impossible to extract any utility from someone else's win. This, in fact, is characteristic of all Hotelling-Downs variants, except for the RCGs that we present here. Furthermore, as opposed to the two-stage solution of [16], abstention is *a priori* a legitimate strategy in RCGs.

In addition, non-atomic voter distribution limits the applicability of HDMs to variants of Plurality voting. In contrast, any rule can be used with our RCG model. For instance, in this paper we study RCGs with general scoring and Condorcet-consistent voting rules.

Appears in: *Proc. of the 16th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2017), S. Das, E. Durfee, K. Larson, M. Winikoff (eds.), May 8–12, 2017, São Paulo, Brazil.*

RCGs are also distinct from the "effective range" HDM variants, as introduced by Feldman et al. [9]. Their idea was to limit the range of attraction of a candidate, so that too distant voters will not vote for her, even if she's the closest candidate. This, effectively, implies that voters may abstain if no candidate has any attraction for them. In contrast, in our work voting is mandatory, i.e., "effective range" is infinite. However, RCGs do include a form of range limitation—the range of positions adoptable by a candidate. Intuitively, an inherently left-wing party cannot reposition itself as extremist right-wing; the fluidity of its political position is limited.

1.1 Contributions

Section 2 introduces the full mathematical detail of our Real Candidacy Games (RCGs) model, that merges Strategic Candidacy Games (SCGs) and Hotelling-Downs models. The section also contains complexity results on the best response computation.

Section 3 contains the initial analysis of our RCG model, and does not limit candidate position choice. The analysis spans several voting rules, and looks into both randomised and lexicographic tie-breaking schemes. Then, Section 4 justifies our choice of closed interval limitation of candidate positioning in the rest of the paper. To achieve this, the section compares the effects of the open-interval and closed-interval limitations.

Section 5 and Section 6 contain the analysis of Nash Equilibrium existence in RCGs under, respectively, randomised and lexicographic tie-breaking schemes. In both cases, multiple voting rules are considered.

1.2 First Example

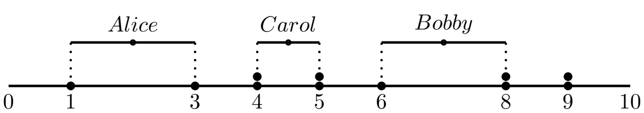

Figure 1: Introductory Example

Before RCG formalities are given, consider the following example. Alice, Carol and Bobby are applying for the same university position. The selection committee has eleven members (henceforth, "voters"), who will elect a candidate once interviews are done. Voters prefer those candidates that are closer to their own research agenda. Let us assume that agendas can be placed along a single axis, e.g., theoretical-experimental. The candidate with the most supporting votes gets the job. If two or more candidates gain the same support, the tie has to be broken, for example via a "lexicographic" ordering based on the number of publications, being the university's graduate, and other objective features. In our example, assume this ordering prefers Alice to Bobby to Carol.

Now, each candidate's research can also be positioned on the voter's agenda axis. However, at the interview it may be presented with an apparent variation, shifting towards a more or less theoretical agenda. Of course this shift is bounded, anchored by factual publications. Figure 1 illustrates our scenario. Voters are depicted as large points on the theoretical-experimental axis, while candidates' *possible* positions are depicted as intervals above the axis.

Bobby can easily position his research to gain the support of 5 voters, and win (possibly due to tie-breaking in his favour). However, prompted by their research agenda similarity, Alice might prefer Carol to win. Alice can achieve this by withdrawing her own application. This would give Carol six supporters, and land Carol the job. Notice that this last situation is stable, i.e., a Nash Equilibrium with respect to participation and positioning of candidates, since no further changes will improve the situation for any of them.

2. MODEL AND PRELIMINARIES

We define our *Real Strategic Candidacy (RCG)* model: Preference Mappings (how candidates affect voter preferences), Voting Rules (guiding voter preference aggregation), and Tie-Breaking Schemes (resolving the ambiguity of equally-preferred candidates). RCGs are then described using specialised forms of these terms.

General Terms. Let there be a set of n voters, $V = \{1, \ldots, n\}$, and a set of m distinct candidates, $C = \{c_1, \ldots, c_m\}$. Voters are characterised by a *voter position* vector $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$. We assume that voters do not strategise and, therefore, voter position is fixed *a priori* as a part of the domain instance. Voter position will be an implicit (omitted) argument in all further definitions of functions and mappings. We generally assume that, w.l.o.g., $p_1 \leq \cdots \leq p_n$. Under the assumption, voters $med_l = \lfloor \frac{n+1}{2} \rfloor$ and $med_r = \lceil \frac{n+1}{2} \rceil$ are *median voters*. If n is odd, we use the notation $med = med_l = med_r$. The set of candidate strategies (positions, states) will be denoted $R_\perp = \mathbb{R} \cup \{\perp\}$, where $\perp$ denotes the possibility of withdrawing from the election. If candidates are prevented from actually adopting the withdrawal strategy, we will explicitly state this, unless the context is clear. For notational convenience both named and indexed notation will be used, so that the *state* (or *strategy*) of a candidate $a = c_i$ may be equivalently denoted $s_a = s_i \in R_\perp$. The overall (joint) *state* will be denoted by the vector $\mathbf{s} = (s_1, \ldots, s_m)$, while $\mathbf{s}_{-i}$ will denote the state vector with the strategy of candidate i removed, so that $\mathbf{s} = (s_i, \mathbf{s}_{-i})$. In addition, we assume that candidates are characterised by *a priori* fixed, *self-supporting* preferences over the set C. That is, each candidate has a preference $\succ_c \in \mathcal{L}(C)$, so that $c \succ_c c'$ for all $c' \neq c \in C$.

Preference Mapping Functions (PMFs) guide the manner in which the position of candidates and voters is translated into voter preferences. That is, while voter position is fixed, their preferences are dynamic. More formally, a PMF maps the candidate joint position, $\mathbf{s}$, to a joint *preference profile*, $(\succ_1, \ldots, \succ_n)$. That is, PMFs have the form $M : \mathbb{R}^m_\perp \to \mathcal{L}(C)^n$, where $\mathcal{L}(C) \subseteq C^m$ is the set of all strict linear orders over the set C. Thus, voter i's preference order is $\succ_i \in \mathcal{L}(C)$, and if i prefers c to c', we write $c \succ_i c'$.

We will be concentrating on PMFs that are consistent with the positional distance, in the following sense. In the context of a joint voter position $\mathbf{p}$ and a joint candidate strategy $\mathbf{s}$, we denote $P = P[\mathbf{s}] : V \cup C \to \mathbb{R}_\perp$ the function that returns the current position of a given voter or candidate, i.e., $P(x) = \begin{cases} p_x & x \in V \\ s_x & x \in C \wedge (s_x \neq \perp) \\ \infty & x \in C \wedge (s_x = \perp) \end{cases}$.

Notice that, as we have underlined before, $P[\mathbf{s}]$ has an additional implicit argument: $\mathbf{p}$. Now, let us define a *distance function* $d_x : V \cup C \to \mathbb{R}$ as $d_x(y) = |P(x) - P(y)|$. A PMF, M, will be consistent with the positional distance of voters and candidates, if for all $i \in V, a, b \in C$ holds that $d_i(a) < d_i(b) \Rightarrow a \succ_i b$, where $(\succ_1, \ldots, \succ_n) = M(\mathbf{s})$. Which, in particular, implies that a withdrawn candidate is placed at the very bottom of a voter's preference order. As a consequence, a withdrawn candidate will not win the election, as long as at least one other candidate participates.[1]

Voting Rules aggregate voter preferences to select the "winning" subset of candidates, and have the formal form of a mapping $\mathcal{F} : \mathcal{L}(C)^n \to 2^C$. A *resolute* voting rule maps preferences into singleton subsets. Otherwise a voting rule is termed *irresolute*. Most popular rules, e.g., Plurality, are naturally irresolute, and require an additional tie breaking scheme (which we discuss later) to determine the unique election's winner. In this paper, unless other-

[1] In a sense, our concept of withdrawal is "incomplete", since voter preferences continue to include the retired candidate.

wise stated, all rules are in the irresolute form.

A **Positional Scoring Rule (PSR)** is defined by a weight vector $\alpha = (\alpha_m, \dots, \alpha_1)$, that modulates the relative importance (or score) of candidates within a preference order. A PSR then accumulates these scores from all voter preference profiles, and selects a candidate with the highest score. More formally, let $(\succ_1, \dots, \succ_n) = \mathcal{M}(\mathbf{s})$, and let $o_i(c, \mathbf{s}) = |\{c' \in C | c \succ_i c'\}| + 1$ be the ranking order of the candidate $c \in C$ with respect to $\succ_i$. The *score* of candidate $c \in C$ is then

$$S_{\mathcal{F}}(c, \mathbf{s}) = \sum_{i=1}^{n} \alpha_{o_i(c,\mathbf{s})},$$

and the overall scoring rule is defined by:

$$\mathcal{F}(\mathcal{M}(\mathbf{s})) = \arg\max_{c \in C} \{S_{\mathcal{F}}(c, \mathbf{s})\}.$$

We shall only be interested in positional scoring rules that are *monotonic*—i.e., that satisfy $\alpha_m \geq \dots \geq \alpha_1$, and assume, w.l.o.g., that $\alpha_1 = 0$. For brevity, we shall henceforth use the term *scoring rule* to describe any monotonic PSR. Well-known examples of such scoring rules are **Plurality** with $\alpha = (1, 0, \dots, 0)$, **Veto** with $\alpha = (1, \dots, 1, 0)$ and **Borda** with $\alpha = (m-1, m-2, \dots, 0)$. The first two are special cases of *k-approval* scoring rules—any scoring rule of the form $\alpha = (1^k, 0^{m-k})$. We call a scoring rule **trivial** if $\alpha = \vec{0}$. Note that if $m = 2$, all non-trivial scoring rules are equivalent to Plurality.

In a **pairwise (sub-)election** between two candidates $a, b \in C$ the Plurality rule is applied, and we say that a **beats** b, if a is ranked first by more candidates than b. Notice that it is possible that $|C| > 2$. A candidate is called a **Condorcet-winner** if she beats every other candidate in a pairwise election. We call a candidate a **Weak Condorcet-winner (WeCo)**, if she is not beaten by any candidate in a pairwise election. It is easy to see that there is at most one Condorcet-winner, but several WeCo's may exist.

A voting rule $\mathcal{F}$ is called **Condorcet-consistent**, if, whenever there exists a candidate c which is a Condorcet-winner,

$$\mathcal{F}(\mathcal{M}(\mathbf{s})) = \{c\}.$$

We shall call the same rule **Super Condorcet-consistent (SCC)**, if, whenever there is at least one WeCo,

$$\mathcal{F}(M(\mathbf{s})) = \{c \in C \mid c \text{ } is \text{ } a \text{ } WeCo\}.$$

Well-known examples for Condorcet-consistent voting rules are *Copeland-0*, *Copeland-1* and *Maximin*, out of which only Copeland-0 is not SCC.[2] Additionally, if there are only two candidates that participate any non-trivial scoring rule is SCC.

2.1 Tie-breaking

Ties may appear both in PMFs, i.e., at the time voter preferences are formed, and in voting rules. In the former case, if two candidates are equidistant from a particular voter, PMF must possess a way to "break the tie" and enforce a strict preference order between these two candidates. Similarly, a way to resolve ambiguity is required when an irresolute voting rule produces a non-trivial set outcome. In this paper, for simplicity, we will assume that the same tie-breaking scheme is used for both tie-event types. In particular, we will be interested in the two classical tie-breakings: the randomised tie-break, which utilises a fair coin toss to resolve ambiguity; and the lexicographic tie-break, that employs a pre-determined full order among candidates.

[2]These rule variants can be found in [11]. We omit them, as we only discuss their supersets: Condorcet-consistent and SCCs.

More formally, the randomised tie-breaking PMF samples $M_{ran} \sim U(\mathcal{M}_{n,m})$, where $\mathcal{M}_{n,m}$ is the set of all valid preference mapping functions over n voters and m candidates, and $U(\cdot)$ is a uniform distribution over a set. The random preference profile $M_{ran}(\mathbf{s})$ produces both orderings of equidistant candidates with the same probability. That is, for every voter i and two distinct candidates $c \neq c'$, if $d_i(c) = d_i(c')$ then

$$Pr(c \succ_i c') = Pr(c' \succ_i c) = \frac{1}{2}.$$

Furthermore, this occurs (pairwise) independently for every distinct triplet (i, c, c'). Notice that the same effect is achieved by sampling M_{ran} uniformly from the set of profiles consistent with $\mathbf{p}$ and $\mathbf{s}$.

Similarly, to resolve the ambiguity of an irresolute voting rule $\mathcal{F}$, we apply a randomly selected choice function on its outcome. Formally, let $\mathcal{F}$ be any voting rule and let $\mathcal{T} \subseteq C^{(2^C)}$ be the set of all choice functions from candidate subsets to a single candidate. Let $\mathcal{V}_{ran} \sim U(\mathcal{T})$ be a uniformly sampled choice function, then $\mathcal{V}_{ran}(\mathcal{F}(\mathcal{P})) \sim U(\mathcal{F}(\mathcal{P}))$ is a randomly tie-broken outcome of rule $\mathcal{F}$ (applied on a voting profile $\mathcal{P}$). Notice that $\mathcal{V}_{ran}(\mathcal{F}(\cdot))$ is a resolute voting rule over preference profiles.

Furthermore, as we assume that PMF and voting rule ties are broken by the same scheme, we can define a resolute voting rule with randomised tie-breaking as well. That is, given a joint candidate strategy $\mathbf{s}$ (and an implicit voter position $\mathbf{p}$), the resolute randomly-broken outcome of the voting rule $\mathcal{F}$ is $\mathcal{V}(\mathbf{s}) = \mathcal{V}_{ran}(\mathcal{F}(M_{ran}(\mathbf{s})))$. For convenience, with slight notation abuse, we shall state $\mathcal{V} = \mathcal{V}_{ran}$ and mean that the randomly-broken voting rule $\mathcal{F}$ is used.

Now, the manner in which we have defined $\mathcal{V}_{ran}(\mathbf{s})$ can be applied to any tie-breaking method to define a resolute form of a given voting rule over the configuration space. In this paper, we will be particularly interested in doing so with another popular tie-breaking schema—*lexicographic tie-breaking*. In more detail, let $\succ_*$ be some fixed linear order over the candidate set C. We define the preference mapping function M_{lex} such that for any voter $i \in V$, and any two candidates c, c' so that $d_i(c) = d_i(c')$ holds that $c \succ_i c' \iff c \succ_* c'$. In addition, we denote $c^* = \max_{\succ_*} C'$, i.e., the unique candidate in $c^* \in C'$ such that for any other $c' \in C'$ $c \succ_* c'$. Thus, given a voting rule $\mathcal{F}$, we define its resolute version with a lexicographic tie-breaking[3] $\mathcal{V}_{lex}(\mathbf{s}) = \max_{\succ_*} \mathcal{F}(M_{lex}(\mathbf{s}))$. Henceforth, when we say that a voting rule $\mathcal{F}$ is used with lexicographic tie-breaking, we refer to $\mathcal{V} = \mathcal{V}_{lex}$.

2.2 Real Candidacy Games

Let us now finalise the formal definition of a *Real Candidacy Game (RCG)*. An RCG is defined as a voting process, based on some voting rule with tie-breaking $\mathcal{V}(\cdot)$, where voter positions are fixed and the candidates are strategic players with action space $\mathbb{R}_\perp$. We will assume that all candidates are selfish and wish to maximise their (expected, in the case of random tie-breaking) utility. In this paper we study several variants of RCG voting, each providing a different additional constraint on the candidate strategy space. In particular, we will consider excluding the withdrawal strategy, or limiting candidate states to intervals. As a notational convenience, we will say that the joint strategy space of candidates is *restricted* to a Cartesian product of intervals, $\mathcal{I} = \{I_i \subseteq \mathbb{R}\}_{i=1}^{m}$, if for all $1 \leq i \leq m$, $s_i \in I_i \cup \{\perp\}$. In particular, if all I_i are closed, we say that a *closed interval restriction (CIR)* is adopted. If $I_i = \mathbb{R}$ for all candidates, then the game is *unrestricted*.

Now, in this paper we will be interested in studying the existence and the properties of a *Pure-strategy Nash Equilibrium (NE)* in RCGs, defined via the concept of *best response* as follows.

[3]$\mathcal{V}_{lex}$ has the same structure as $\mathcal{V}_{ran}$, since $\max_{\succ_*}(\cdot) \in \mathcal{T}$.

Let $\mathcal{V} = \mathcal{V}_{lex}$ be a voting rule with lexicographic tie-breaking. A *best response* of a candidate $c \in C$ to a partial joint state s_{-c} is a position $s_c \in I_c$ such that for any $s'_c \neq s_c \in I_c$ holds that

$$\mathcal{V}(s_c, s_{-c}) \succeq_c \mathcal{V}(s'_c, s_{-c}).$$

Similarly, let $\mathcal{V} = \mathcal{V}_{ran}$ be a rule with randomised tie-breaking, and let there be a preference-consistent utility function $u_c : C \to \mathbb{R}$ associated with each voter, i.e., for any two candidates $a, b \in C$

$$u_c(a) > u_c(b) \Rightarrow a \succ_c b.$$

Then, a best response of a candidate $c \in C$ to s_{-c} is a strategy $s_c \in I_c$ so that for all s'_c

$$\mathbb{E}[u_c(\mathcal{V}(s_c, s_{-c}))] \geq \mathbb{E}[u_c(\mathcal{V}(s'_c, s_{-c}))],$$

where the expectation is taken over tie-breaking decisions. In both cases, we will denote the set of all best responses[4] of candidate c by $\mathcal{B}_c(\mathbf{s}_{-c})$. Finally, we say that a joint state $\mathbf{s}$ is a *Pure-strategy Nash Equilibrium (NE)*, if for all candidates $c \in C$ holds $s_c \in \mathcal{B}_c(\mathbf{s}_{-c})$.

As we have mentioned, in this paper we will only be concerned with the existence and features of a NE under different RCG strategy constraints, leaving the issue of finding NEs for future work. However, as a preliminary step in this direction we provide the following feasibility result on computing the best response, suggesting that at least best-response iteration algorithms are a feasible option.

Proposition 1. *Let $\mathcal{F}$ be a voting rule, computable in $O(T_{n,m})$ time for any preference profile of n voters over m candidates, where $T_{n,m}$ is some function of n and m. Let $\mathcal{V} = \mathcal{V}_{lex}$ be the lexicographically tie-broken version of $\mathcal{F}$. Then for any candidate $c \in C$, the best response set, $\mathcal{B}_c(\mathbf{s}_{-c})$, is computable in*

$$O(n \cdot m \cdot [T_{n,m} + \log(m)]).$$

In particular, if T is a polynomial function, then computing $\mathcal{B}_c(p, s)$ can be done in polynomial time.

The proof of Proposition 1 relies on our ability to break a candidate's strategy interval I_c into a polynomial number of sub-intervals, so that all strategies s_c within a sub-interval lead to the same election outcome in response to $\mathbf{s}_{-c}$. As a result, as long as a single voting rule query takes polynomial time, the best response strategy can be efficiently calculated by direct evaluation of all sub-interval outcomes. Unfortunately, the same method cannot be applied to a voting rule with randomised tie-breaking. Unlike lexicographic, randomised tie-breaking does not consistently produce the same outcome. In fact, in the worst case, the number of "black-box" queries to the voting rule, necessary to evaluate the utility of a candidate's strategy, can be as high as $(m!)^n$. However, although computing the exact best response may be difficult, for some voting rules it is possible to check whether a given candidate has a non-zero chance of winning the election.

Proposition 2. *Consider an RCG with n voters and m candidates, constructed over the k-approval scoring rule with random tie-breaking for some $k \in \{1, \ldots, m-1\}$. Then for any joint candidate strategy $s \in \mathbb{R}^m_\perp$ and a candidate $c \in C$, determining whether $Pr(\mathcal{V}(s) = c) > 0$ takes $O(poly(n, m))$ time.*

In fact, it is possible to compute all possible effects of a candidate's response strategy on the set of possible winners.

Corollary 1. *Let $c \in C$ be some candidate, and $\mathbf{s}_{-c}$ a partial joint strategy of her opponents, and let $\mathcal{V}$ be the k-approval scoring rule with lexicographic tie-breaking. Define the* possible winner *mapping $PW \subset I_c \times 2^C$ so that*

$$(s_c, \Omega) \in PW \iff \Omega = \{c \in C | P(\mathcal{V}(s_c, \mathbf{s}_{-c})) > 0\}.$$

Then, the mapping PW is $O(poly(n, m))$ time computable.

3. UNRESTRICTED STRATEGIES

We begin our investigation of RCGs by assuming unrestricted candidate strategies, and consider RCGs based on scoring and Condorcet-consistent voting rules with both lexicographic and random tie-breaking schemes. Such RCGs are a convenient set of models to demonstrate basic principles of RCG analysis. They are, however, not without real-world parallels. For instance, the model in Subsection 3.1 is descriptive of a pseudo-democracy in which the governing party has an obvious tie-breaking advantage. The only reason elections are held by such "democracies" is to show that the party has "its hand on the public's pulse". In turn, Subsection 3.2 addresses the situation that arises in political systems with a very high "turnover" rate of parties. None of the new parties are bound in their strategy, except by the current popular trend, and there are many of them. As a result, votes are cast almost randomly.

3.1 Lexicographic Tie-breaking

As one would expect from our "pseudo-democracy" example, the only possible winner in a NE of an RCG with unrestricted strategies is the top-ranking candidate of the tie-breaking order, $c^* = \max_{\succ_*} C$. In what follows, we prove this formally for RCGs based on scoring and SCC voting rules.

We begin with the following useful lemma that relates the distance of a candidate from median voters to its electoral strength.

Lemma 1. *Let an RCG be based on the lexicographic tie-breaking order $\succ_*$, and a voter position vector $\mathbf{p} \in \mathbb{R}^n$. In addition, for a joint candidate strategy $\mathbf{s} \in \mathbb{R}^m_\perp \setminus \{\perp^m\}$, let*

$$w_l = \max_{\succ_*} \arg\min_{c \in C} \{d_{med_l}(s_c)\},$$
$$w_r = \max_{\succ_*} \arg\min_{c \in C} \{d_{med_r}(s_c)\}.$$

Then w_l and w_r are WeCos. Furthermore, if $P(med_l) = P(med_r)$, then $w = w_l = w_r$ is a Condorcet-winner.

Lemma 1 is a key tool throughout the paper, though simple enough to omit its proof. Proposition 3 is the first of many to rely on it.

Proposition 3. *Let an RCG be based on a voting rule, $\mathcal{V} = \mathcal{V}_{lex}$, that is either Condorcet-consistent with $P(med_l) = P(med_r)$ or SCC, and the tie-breaking order $\succ_*$. If candidate strategies are unrestricted, then $\mathcal{V}(s) = c^*$ for any NE strategy $\mathbf{s} \in \mathbb{R}^m_\perp$.*

Intuitively, Proposition 3 shows that, without further restrictions, lexicographic tie-breaking order gives absolute power to c^*. Interestingly, this is also true for a dramatically different class of voting rules, monotonic PSRs.

Proposition 4. *Let an RCG be based on $\mathcal{V} = \mathcal{V}_{lex}$, a monotonic PSR, lexicographically tie-broken by $\succ_*$. If candidate strategies are unrestricted, then $\mathcal{V}(\mathbf{s}) = c^*$ for any NE strategy $\mathbf{s}$.*

PROOF. Let $\mathbf{s} \neq \perp^m$ be a NE, and let us assume that $\mathcal{V}(s) \neq c^*$. Denote by s_i^{∞} and $s_i^{-\infty}$ strategies of candidate i so that

$$s_i^{\infty} > \max_{j \in V \cup C \setminus \{i\}} s_j + k \cdot \max_{k,l \in V \cup C \setminus \{i\}} |s_k - s_j|$$
$$s_i^{-\infty} < \min_{j \in V \cup C \setminus \{i\}} s_j - k \cdot \max_{k,l \in V \cup C \setminus \{i\}} |s_k - s_j|$$

[4]Notice that the set is implicitly dependent on voter positions.

where k is sufficiently large. That is, s_i^{∞} (respectively, $s_i^{-\infty}$) is a strategy of candidate i that is, comparatively, much farther to the right (respectively, left) than any other candidate or voter.

Now, let $\mathbf{s}' = (s_{c^*}^{\infty}, s_{-c^*})$. Notice that $\mathcal{V}(s') = c \neq c^*$, otherwise s is not an NE, for c^* can improve its utility by adopting $s_{c^*}^{\infty}$. In addition, let us define a joint strategy $\mathbf{s}^* = (s_{c^*}^*, \mathbf{s}_{-c^*})$, where $s_{c^*}^* = s_c$. In other words, two modifications of the joint strategy $\mathbf{s}$, where c^* has shifted either very far to the right, or to the position of the winning candidate $\mathcal{V}(\mathbf{s}')$. Notice that c^* is ranked the lowest by all voters, when candidates adopt $\mathbf{s}'$. Thus $S_{\mathcal{F}}(c^*, \mathbf{s}') = 0$, while $\mathcal{V}(\mathbf{s}') = c$ and $S_{\mathcal{F}}(c, \mathbf{s}') = \max_{c' \in C} \{S_{\mathcal{F}}(c', \mathbf{s}')\}$.

On the other hand, adopting $\mathbf{s}^* = (s_{c^*}^*, \mathbf{s}_{-c^*})$, candidates create a voter preference profile where $c^* \succ_i c$. This is because c^* and c are tied and $c^* \succ_* c$. At the same time, the relative ranking between the candidate c and all other candidates in $C \setminus \{c^*\}$ is independent of whether $\mathbf{s}^*$ or $\mathbf{s}'$ is adopted. Thus for all voters $o_i(c^*, \mathbf{s}^*) \geq o_i(c, \mathbf{s}')$, and $S_{\mathcal{F}}(c^*, \mathbf{s}^*) \geq S_{\mathcal{F}}(c, \mathbf{s}')$.

Now, when c^* changes its strategy from $s_{c^*}^{\infty}$ in $\mathbf{s}'$ to $\mathbf{s}_{c^*}^*$ in $\mathbf{s}^*$, voters increase its rank, but did not increase the rank of all other candidates. Hence $\forall c' \neq c^*$, $\forall i \in V$, $o_i(c', \mathbf{s}') \geq o_i(c', \mathbf{s}^*)$, which implies that $S_{\mathcal{F}}(c', \mathbf{s}') \geq S_{\mathcal{F}}(c', \mathbf{s}^*)$ and $S_{\mathcal{F}}(c^*, \mathbf{s}^*) \geq S_{\mathcal{F}}(c', \mathbf{s}^*)$. Therefore, $\mathcal{V}(\mathbf{s}^*) = c^*$, implying that $s_{c^*} \notin \mathcal{B}_{c^*}(\mathbf{s}_{-c^*})$, i.e., not a best response, contradicting the assumption that $\mathbf{s}$ is a NE. □

3.2 Random Tie-breaking

The situation is somewhat more "liberal" under random tie-breaking, since, as we show in this section, every candidate can "better respond" to guarantee a positive probability of winning.

Proposition 5. *Let a RCG be based on a voting rule, $\mathcal{V} = \mathcal{V}_{ran}$, that is either Condorcet-consistent with $P(med_l) = P(med_r)$, SCC or a monotonic PSR. For all $\mathbf{s} \in \mathbb{R}_{\perp}^m$ and any candidate $c \in C$, there is $s'_c \in \mathbb{R}$ such that $Pr(\mathcal{V}(s'_c, \mathbf{s}_{-c}) = c) > 0$.*

PROOF. Let $M_{\succ_c}$ be a lexicographic tie-breaking PMF with respect to $\succ_c$. If $M_{ran} = M_{\succ_c}$ then according to Propositions 3 and 4, there is $s'_c \in \mathbb{R}$ such that $\mathcal{V}(s'_c, \mathbf{s}_{-c}) = c$. We have

$$Pr(M_{ran} = M_{\succ_c}) > 0 \;\;\Rightarrow\;\; Pr(\mathcal{V}(s'_c, \mathbf{s}_{-c}) = c) > 0,$$

since $M_{\succ_c} \in \mathcal{M}_{n,m}$. □

Furthermore, for some forms of the utility function, an NE strategy results in a lottery where every candidate has a chance to win.

Corollary 2. *Under the same conditions as in Proposition 5, if for all $c, c' \in C$,*

$$u_c(c') = \begin{cases} 1 & if\ c' = c \\ 0 & otherwise \end{cases}$$

then for any NE strategy $\mathbf{s}$, holds $Pr(\mathcal{V}(\mathbf{s}) = c) > 0$ for all $c \in C$.

To conclude this section, we note that, if strategies are unrestricted, equilibrium properties are strongly determined by the tie-breaking order, with very little regard for voter and candidate positions. In addition, as real-world political elections show, candidates are expected to respect some boundaries in their strategy. For example, to maintain credibility, certain statements on taxation might be expected from a Democratic candidate.

4. STRATEGY RESTRICTION CHOICE

As we have seen in Section 3, using unrestricted strategy space has very little strategic depth. Rather, it is the tie-breaking rule that dictates the equilibria. We, therefore, accept an interval restriction on the strategy space. However, it is yet unclear whether closed or open intervals should be used. Following our political election intuition: politicians are required to place clear "red-line" boundaries on their position, boundaries they may adopt, but never cross.

As we will show in Propositions 6 and 7, open intervals may introduce strategic instability, i.e., it is possible that no NE exists.

We begin by showing that open interval RCGs may lack NEs, if they are based on a certain class of voting rules.

Definition 1 (UNANIMITY). *A candidate c is the unanimous top choice, if $c \succ_i c'$ for any voter $i \in V$ and a candidate $c' \neq c \in C$. A voting rules $\mathcal{F}$ satisfies* **unanimity**, *if $\mathcal{F}(\mathcal{M}(\mathbf{s})) = \{c\}$, whenever c is the unanimous top choice under $\mathcal{M}(\mathbf{s})$.*

Proposition 6. *Let $\mathcal{V} = \mathcal{V}_{lex}$ be based on an irresolute rule $\mathcal{F}$. If $\mathcal{F}$ satisfies unanimity, then, for any $p \in \mathbb{R}^n$ and any number of candidates $m \geq 2$, there is an RCG instance with open interval strategy space $\mathcal{I}$, which has no NE.*

On the other hand, to destabilise RCGs based on $\mathcal{V}_{ran}$ an additional feature of utility functions is necessary.

Definition 2 (UNCOMPROMISING UTILITIES). *We call an ordered set of utility functions* uncompromising *if there is no* compromise candidate *$c \in C$, so that*

$$\forall i \in \{1, \ldots, m\},\; u_i(c) = \max_{c' \in C} \{u_i(c')\}$$

Equivalently, $\forall c \in C,\; \exists c' \in C,\;\; u_{c'}(c') > u_{c'}(c)$.

Remark 1. *Existence of a compromise candidate c guarantees NE existence, if withdrawals are allowed. For example, $\mathbf{s} = (s_c, \mathbf{s}_{-c})$, where only $s_c \neq \perp$, is a NE.*

Proposition 7. *Let $\mathcal{V} = \mathcal{V}_{ran}$ with uncompromising utilities be based on an irresolute rule $\mathcal{F}$. If $\mathcal{F}$ satisfies unanimity, then for all $p \in \mathbb{R}^n$ and any number of candidates $m \geq 2$, there is an RCG instance with open interval strategy space $\mathcal{I}$, which has no NE.*

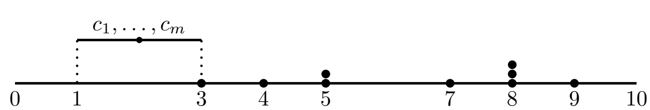

Figure 2: Illustration of the proof of Proposition 7.

PROOF. Assume w.l.o.g. $p_1 = \min\{p_i\}$ and for all $c \in C$ set $I_c = (p_1 - 1, p_1)$. Let $\mathbf{s} \neq \perp^m$ be some joint strategy, and let

$$c \in \arg\max_{c' \in C} \{s_{c'}\}.$$

Now, $Pr(\mathcal{V}(\mathbf{s}) = c) > 0$, since c has a positive probability to be the unanimous top-choice, and $\mathcal{F}$ satisfies unanimity. In turn, uncompromising utilities imply that a candidate $c' \neq c$ exists so that $u_{c'}(c') > u_{c'}(c)$, and $\mathbb{E}[u_{c'}(\mathcal{V}(s))] < u_{c'}(c')$. However, for small enough $\epsilon > 0$, if c' moves to $s_c + \epsilon \in I_{c'}$, she guarantees victory and the expected utility of $u_{c'}(c')$. Hence, $\mathbf{s}$ is not a NE. □

Although this point is a matter of discussion, lack of an equilibrium is commonly associated with negative instability of political elections. Therefore, seeking stability, we may need to assume *closed interval* restrictions on candidate strategies. Intuitively, this requires that political views of a candidate have clear-cut, adoptable boundaries, without "grey area issues". Still, *a priori*, it is unclear whether a closed interval restriction alone is sufficient to guarantee NE existence. Sections 5 and 6 investigate this matter in detail.

5. LEXICOGRAPHIC TIE-BREAKING

Here we adopt the closed-interval restriction (CIR), and study the existence of NEs in RCGs with lexicographic tie-breaking. We find that under CIR, the voting rule at the base of an RCG plays a far more significant role, than without CIR. For example, we find a set of simple conditions that guarantee NE existence under Condorcet-consistent and SCC rules. On the other hand, under PSRs arbitrarily large RCGs may exist without a NE, even though similarly generic sufficient NE existence conditions exist.

5.1 Condorcet-consistent Voting Rules

Sufficient conditions, for NE existence under SCC and Condorcet-consistent rules, will rely on some select candidate properties.

Definition 3. **Noteworthy candidates** *are members of the following candidate subsets.*

1. *Potentially closest candidates to med_l and med_r with the highest ranking in the tie-breaking order, denoted l_1 and r_1:*

$$l_1 = \max_{\succ_*}\left[\arg\min_{c\in C}\left\{\min_{x\in I_c}|P(med_l)-x|\right\}\right]$$

$$r_1 = \max_{\succ_*}\left[\arg\min_{c\in C}\left\{\min_{x\in I_c}|P(med_r)-x|\right\}\right]$$

2. *The set of* middle candidates *MID:*

$$MID = \{c \in C\backslash\{l_1, r_1\} \mid I_c \cap [P(med_l), P(med_r)] \neq \emptyset\}$$

Definition 4. *Given a joint strategy $\mathbf{s}$, and the induced candidate orders $(\succ_1, \ldots, \succ_n)$. We say that $c \in C$* **covers** *med_l if for all $c' \in C$ holds $\mathbf{s}_{c'} \in \mathbb{R} \Rightarrow c \succ_{med_l} c'$, and no c' with $\mathbf{s}_{c'} = \bot$ can alter its strategy to reduce med_l's ranking of c. If c is ranked top, and no candidate at all can alter its strategy to reduce med_l's ranking of c, we say that c* **secures** *med_l. Symmetrically, we define c covering/securing of med_r.*

Notice that Definitions 3, 4 imply that only l_1 can secure med_l, and only r_1 can secure med_r.

Definition 5. *Given a joint strategy $\mathbf{s}$, a candidate $c \in C$ is a* **Guaranteed Weak Condorcet-winner (GWeCo)***, if c is a WeCo w.r.t. $\mathcal{M}(\mathbf{s})$, and remains a WeCo w.r.t. $\mathcal{M}(\mathbf{s}')$ for any $\mathbf{s}' = (s_c, \mathbf{s}'_{-c})$, where $\mathbf{s}'_{-c} \in \mathbb{R}_{\bot}^{m-1}$.*

Note that GWeCos are necessarily either members of MID, when placed anywhere between the medians, or l_1 and r_1, when they secure med_l and med_r. With these definitions in place, we are now ready to state and prove the following proposition.

Proposition 8. *Let an RCG be based on $\mathcal{V} = \mathcal{V}_{lex}$ with a Condorcet-consistent voting rule. If $P(med_l) = P(med_r)$, in particular if the number of voters is odd, then there is always a NE. Furthermore, for any NE $\mathbf{s}$ holds $l_1 = r_1$ and $\mathcal{V}(\mathbf{s}) = \{l_1\}$.*

PROOF. It is easy to see that $l_1 = r_1$, if $P(med_l) = P(med_r)$. Let $\mathbf{s} \in \mathbb{R}^m$ be such that l_1 secures both medians and all other candidates participate. Conditions of Lemma 1 are satisfied, and $w_l = l_1$ in that context, making it the winner of the election. Since s_{l_1} minimises the distance to med_l, l_1 is also a GWeCo and will continue to win the elections regardless of the other candidates' strategic shifts (even withdrawals). Therefore, $\mathbf{s}$ is a NE.

Now let $\mathbf{s}'$ be any other NE. l_1 can always secure the medians and win, by shifting $\mathbf{s}'_{l_1}$ as close as possible to $P(med_l)$. As $s'_{l_1} \in \mathcal{B}(\mathbf{s}'_{-l_1})$ must hold, it must be that $\mathcal{V}(\mathbf{s}') = l_1$ as well. $\square$

Interestingly, under SCC voting rules, NE existence holds under far more general conditions, as we show in Proposition 9.

Proposition 9. *Let an RCG be based on $\mathcal{V} = \mathcal{V}_{lex}$ with a SCC rule. NE exists, if withdrawals are allowed within the RCG.*

We omit the complete proof of Proposition 9, and in lieu of a proof sketch give here the Lemma on which it relies.

Lemma 2. *Let $\mathbf{s}$ be a joint candidate strategy. Let a and b be two candidates that, under $\mathbf{s}$, cover respectively med_l and med_r. Then any candidate $c \in C \setminus MID$, so that $s_c = \bot$, can neither become a WeCo, nor change the election winner, by altering its strategy to some $\mathbf{s}'_c \in \mathbb{R}$.*

Now, it must be noted that NE features, shown under Proposition 8 conditions, do not necessarily hold under conditions of Proposition 9. Furthermore, NE strategies are not simply about minimising the distance from median voters. Consider an RCG instance consistent with Example 1 below.

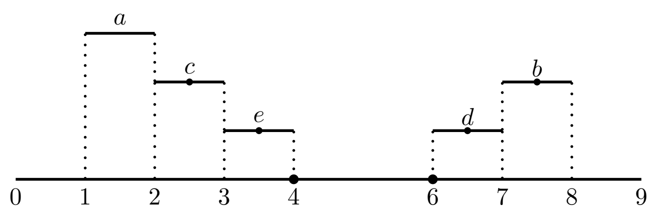

Figure 3: Illustration of Example 1.

Example 1. *Let the set of voters be $V = \{1, 2\}$ and the set of candidates $C = \{a, b, c, d, e\}$. Voter positions are $p = (4, 6)$ and candidates adopt CIR, where $I_a = [1, 2]$, $I_c = [2, 3]$, $I_e = [3, 4]$, $I_d = [6, 7]$ and $I_b = [7, 8]$. The lexicographic tie-breaking order is $a \succ_* b \succ_* c \succ_* d \succ_* e$, and candidate preferences must include the following: $a \succ_c b$, $b \succ_d c$ and $c \succ_e b \succ_e a \succ_e d$. Since there is always at least one WeCo, all SCC voting rules are equivalent.*

Now, let us have a closer look at the joint strategy

$$\mathbf{s} = (s_a, s_b, s_c, s_d, s_e) = (\bot, \bot, 3, 6, \bot),$$

i.e., only candidates c and d participate positing themselves as close as possible to voters. It is easy to see that this is a NE strategy. However, under $\mathbf{s}$, $l_1 = e$ and $r_1 = d$, but $\mathcal{V}(\mathbf{s}) = c \neq l_1, r_1$. Furthermore, some strategies that minimise the distance to median voters are not NEs at all, and even lead to best response cycles. For example, let us inspect the joint strategy:

$$\mathbf{s}' = (s_a, s_c, s_e, s_d, s_b) = (2, 3, 4, 6, 7).$$

Under $\mathbf{s}'$ the winner is $\mathcal{V}(\mathbf{s}') = d$. However, candidate e prefers c to d, and, in fact, can make c the winner if e withdraws (or moves to position 3). In particular, $\mathbf{s}'$ is not a NE. Furthermore, withdrawals form the following cycle of best responses. Each joint strategy in this cycle is denoted by the set of participating voters, and implicit positions coincide with those in $\mathbf{s}'$.

$$\{a, c, d, b\} \to \{a, c, b\} \to \{a, b\} \to \{a, e, b\}$$
$$\to \{a, e, d, b\} \to \{a, d, b\} \to \{a, c, d, b\}$$

5.2 Monotonic Scoring Rules

Changing to RCGs based on PSRs brings another layer of strategic complexity. In particular, while some PSR sub-families readily support NE existence, there are cases where arbitrarily large RCGs exist without an NE.

Proposition 10. *Let an RCG be based on $\mathcal{V} = \mathcal{V}_{lex}$ with a monotonic PSR. If withdrawals are allowed within the RCG and $P(med_l) = P(med_r)$, e.g., when the number of voters is odd, then a NE exists.*

To see whether a sufficient NE-existence condition can cover a wider range of candidate sets than Proposition 10, we first turn to more specialised PSRs, namely Plurality.

Proposition 11. *Let an RCG be based on $\mathcal{V} = V_{lex}$ with Plurality. If $m = 2$, then a NE exists.*

However, starting from $m = 3$, subtler conditions may be needed. To show this, we rely on the following RCG scenario.

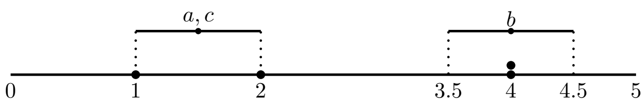

Figure 4: Illustration of Example 2.

Example 2. *Let the set of voters be $V = \{1, 2, 3, 4\}$ and the set of candidates $C = \{a, b, c\}$. Voter positions are $p = (1, 2, 4, 4)$ and candidates may choose positions within their respective intervals: $I_a = I_c = [1, 2]$ and $I_b = [3.5, 4.5]$. The lexicographic tie-breaking is subject to $a \succ_* b \succ_* c$ and we assume only that $b \succ_c a$.*

Proposition 12. *If $m \geq 3$ and withdrawals are not allowed, then RCG instances exist, based on $\mathcal{V} = \mathcal{V}_{lex}$ over Plurality, without any NE.*

PROOF. Notice that the RCG scenario of Example 2 satisfies the proposition conditions. We will now show that there is no NE strategy possible. First, note that candidate b is guaranteed to get votes 3 and 4 under any $\mathbf{s} \in \mathbb{R}^m$, and that candidate c prefers candidate b to win over a. Now, let us assume that some $\mathbf{s} \in \mathbb{R}^m$ is a NE. If $s_a \neq s_c$ then a and c only get one vote each and b wins. However, in this case, a has a beneficial deviation to $s'_a = s_c$, which would make it a winner: voters would tie break in a's favour, granting it two votes, and the same tie-breaking order will prefer a to b with the same number of votes. Thus, for $\mathbf{s}$ to be a NE, it must hold that $s_a = s_c$. Alas, in this case c has a beneficial deviation—it may move away from a, making b the winner, which she prefers. Therefore, s cannot be a NE.

The scenario can be extended to $m > 3$ by placing additional candidates at $s_i = 5$, where they will get no votes at all. $\square$

Interestingly, $m = 3$ is a pivotal number. For $m = 3$, allowing withdrawals is sufficient for NE existence, while for $m \geq 4$ it is no longer enough. The latter claim relies on introducing an additional candidate into Example 2.

Proposition 13. *Let an RCG be based on $\mathcal{V} = \mathcal{V}_{lex}$ over Plurality, and let $m = 3$. Then, if withdrawals are allowed in the RCG instance, a NE exists.*

Proposition 14. *If $m \geq 4$, then RCG instances exist, based on $\mathcal{V} = \mathcal{V}_{lex}$ over Plurality, both with and without allowing withdrawals, without any NE.*

6. RANDOM TIE-BREAKING

In this section we continue to adopt the closed-interval restriction (CIR), but concentrate on RCGs with random tie-breaking. We provide RCG scenarios with and without NEs, and contrast them with the results of Section 5.

6.1 Condorcet-consistent Voting Rules

RCG scenarios based on Condorcet-consistent rules, that have a NE, are quite ubiquitous and easy to construct. However, unlike the case with lexicographic tie-breaking, there is no guarantee of NE existence if the number of voters is odd. Furthermore, no such guarantee can be given for RCGs based on SCC rules, when withdrawals are allowed.

Proposition 15. *There is a SCC voting rules, $\mathcal{F}$, so that RCG instances exist, based on $\mathcal{V} = \mathcal{V}_{ran}$ over $\mathcal{F}$, without a NE. Furthermore, such instances may have an odd number of candidates or withdrawals allowed.*

The proof of Proposition 15 is based on the following scenario.

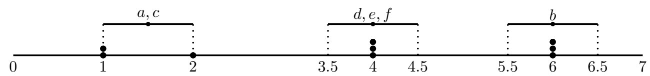

Figure 5: Illustration of Example 3.

Example 3. *Let the set of voters be $V = \{1, \ldots, 9\}$ and the set of candidates $C = \{a, b, c, d, e, f\}$. Let voter position be $\mathbf{p} = (1, 1, 2, 4, 4, 4, 6, 6, 6)$ and candidates adopt CIR, where $I_a = I_c = [1, 2]$, $I_d = I_e = I_f = [3.5, 4.5]$ and $I_b = [5.5, 6.5]$. Let $\mathcal{F}$ be the voting rule that returns all WeCos, if such exist, and otherwise returns the sole winner of a Plurality election with lexicographic tie-breaking order $a \succ_* b \succ_* c \succ_* d \succ_* e \succ_* f$. Candidate preferences are such that the following utility functions are consistent with them:*

$$\forall x, y \in C, \;\; u_y(x) = \begin{cases} 1 & y = c \wedge (x = c \vee x = b) \\ 1 & y \neq c \wedge x = y \\ 0 & otherwise \end{cases}$$

In other words, candidate c draws positive utility from either herself or candidate b winning, while other candidates are purely selfish.

Notice that RCG scenarios of Example 3 indeed have no NE, even if candidates can withdraw.

6.2 Monotonic Scoring Rules

RCG scenarios based on a monotonic PSR, that have a NE, are also easy to come by. As a representative of a relevant scenario construction, consider Example 4 with respect to the Veto rule.

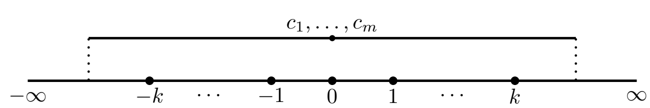

Figure 6: Illustration of Example 4.

Example 4. *Let there be a set V of $2k + 1$ voters, positioned at $\mathbf{p} = (-k, -k + 1, \ldots, k)$, and a set of m candidates $C = \{c_1, \ldots, c_m\}$. Candidates adopt CIR, and $I_j = [-l \cdot k, l \cdot k]$ for all $c_j \in C$, where $l >> 10$ is sufficiently large. In turn, candidate utility functions are purely selfish, i.e.,*

$$\forall c, c' \in C \;\; u_c(c') = \begin{cases} 1 & c = c' \\ 0 & otherwise \end{cases}$$

If an RCG consistent with the scenario of Example 4 is based on the Veto rule, then the joint strategy $\mathbf{s}$, where $s_c = 0$ for all $c \in C$,

Voting Rule	Withdrawals	Single Median	#Candidates	Lex	Ran
SCC	Yes	Any	Any	✓	×
SCC	No	Any	Any	✓?	×
Condorcet-consistent	Any	Yes	Any	✓	×
Monotonic scoring rule	Yes	Yes	Any	✓	×?
Plurality	Any	Any	2	✓	✓?
Plurality	Yes	Any	3	✓	?
Plurality	No	No	3	×	?
Plurality	Any	No	≥ 4	×	×

Table 1: Conditions of an equilibrium existence: Summary of results and open questions.

is a NE. To see this, consider a candidate $c \in C$ that attempts to deviate from $\mathbf{s}$. In this case, $c \in C$ is vetoed by at least $k+1$ voters, while all other candidates receive only k vetoes. Thus c may never become the election winner. On the other hand, if c abides by $\mathbf{s}$, it has the same positive probability of winning as any other candidate. Thus $\mathbf{s} = \vec{0}$ is a NE.

Unfortunately, for some monotonic PSRs, examples of random tie-breaking RCGs without a NE can be constructed. Furthermore, they are sufficiently generic to be modified for any number of candidates, as was also the case with lexicographic tie-breaking RCGs.

Proposition 16. *If $m \geq 4$, then RCG instances exist, based on $\mathcal{V} = \mathcal{V}_{ran}$ over Plurality, both with and without allowing withdrawals, that have no NE.*

The proof of this proposition relies on the following scenario.

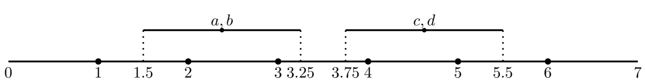

Figure 7: Illustration of Example 5.

Example 5. *Let there be a set V of 6 voters, positioned at $\mathbf{p} = (1, 2, \ldots, 6)$, and a set of 4 candidates, $C = \{a, b, c, d\}$. Let candidates adopt CIR, with strategic intervals $I_a = I_b = [1.5, 3.25]$ and $I_c = I_d = [3.75, 5.5]$. Let candidate utility functions be purely selfish, i.e.,*

$$\forall x, y \in C, \;\; u_x(y) = \begin{cases} 1 & x = y \\ 0 & otherwise \end{cases}$$

Notice that RCGs, consistent with the scenario of Example 5 and based on $\mathcal{V} = \mathcal{V}_{ran}$ over Plurality, indeed have no NE, whether candidates are allowed to withdraw or not.

7. CONCLUSIONS

In this paper we have introduced a novel model of strategic candidacy—Real Candidacy Games (RCGs). RCGs combine features of the Hotelling-Downs model (a continuous, but possibly limited, range of motion for candidates) with classical Strategic Candidacy Games (where candidates have their own, non-trivial preference over the candidate set, and an atomic distribution of voters). We explored RCG features, such as the existence of a Nash Equilibrium (NE), in scenarios based on various voting rules and strategic restrictions. In particular, we have considered the effects of adopting both lexicographic and random tie-breaking, using Condorcet-consistent and positional scoring rules, as well as placing interval restrictions on candidate strategies or disallowing withdrawal.

We show that, if candidate strategies are unrestricted, the tie-breaking method has greater effect on the election outcome than the specific voting rule in place. For example, we show that lexicographic tie-breaking guarantees that a particular candidate will win, while randomised tie-breaking allows all candidates to have positive probability of winning. However, if interval strategy restrictions are placed, RCGs gain strategic depth, and are no longer resolved trivially. For example, we show that a combination of lexicographic tie-breaking and a (Super) Condorcet-consistent rule is guaranteed to have a NE. Yet, when the rule is replaced by a scoring rule, arbitrarily large RCG examples can be constructed that have no NE. Nonetheless, some guarantees of NE existence can be provided for the use of scoring rules as well. Interestingly, these guarantees may vanish and be replaced by others, when random tie-breaking is used, in place of the lexicographic one.

Our results are summarised in Table 1, where check-marks point to conditions that guarantee the existence of a pure NE in RCGs, cross-marks denote existence of a counter-example (a RCG without a NE), and question marks naturally indicate open questions.

Now, in spite of the great range of results that we obtain, there are several directions in which the RCG model can be further studied and extended. First, although we do show that the best response can be poly-time computable, it is yet unclear whether the same would hold for a NE. In fact, most computational and algorithmic questions remain open. RCGs with random tie-breaking prove to be particularly intriguing in this respect.

Our model also contains several behavioural assumptions that we plan to lift. For example, rather than just being an instrument in a game between candidates, voters should be enabled to act strategically as well. More immediate, however, is the need to investigate more complex candidate utility functions and more general voting rules. For example, we may address the question of strategic candidacy games in parliamentary elections from the RCG point of view. That is, through a combination of a set-oriented utility function and a non-resolute voting rule, proportional representation elections with strategic candidates can be investigated.

Finally, it is necessary to deal with more general strategy space topologies, for politics are rarely single-dimensional. Rather the political spectrum runs across multiple issues, and will necessitate an extension of our model to multi-issue voting schemes. In addition, looking back at the origins of Hotelling-Downs, and considering global economies, it will also be necessary to extend RCGs to non-trivial topologies, like cycles and spheres.

Acknowledgment

This research has been partly funded by Israel Science Foundation grant #1227/12, and by the Israeli Center of Research Excellence in Algorithms (I-CORE-ALGO). While performing parts this research S. Obraztsova was affiliated with I-CORE-ALGO at Hebrew University.

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